

Picard curves with small conductor

Michel Börner, Irene I. Bouw, and Stefan Wewers

Abstract We study the conductor of Picard curves over \mathbb{Q} , which is a product of local factors. Our results are based on previous results on stable reduction of superelliptic curves that allow to compute the conductor exponent f_p at the primes p of bad reduction. A careful analysis of the possibilities of the stable reduction at p yields restrictions on the conductor exponent f_p . We prove that Picard curves over \mathbb{Q} always have bad reduction at $p = 3$, with $f_3 \geq 4$. As an application we discuss the question of finding Picard curves with small conductor.

Key words: 2010 *Mathematics Subject Classification*. Primary 14H25. Secondary: 11G30, 14H45.

1 Introduction

Let Y be a smooth projective curve of genus g over a number field K . To simplify the exposition, let us assume that $K = \mathbb{Q}$. With Y we can associate an L -function $L(Y, s)$ and a conductor $N_Y \in \mathbb{N}$. Conjecturally, the L -function satisfies a functional equation of the form

$$\Lambda(Y, s) = \pm \Lambda(Y, 2 - s),$$

where

$$\Lambda(Y, s) := \sqrt{N_Y}^s \cdot (2\pi)^{-gs} \cdot \Gamma(s)^g \cdot L(Y, s).$$

By definition, both $L(Y, s)$ and N_Y are a product of local factors. In this paper we are really only concerned with the conductor, which can be written as

$$N_Y = \prod_p p^{f_p}.$$

Institut für Reine Mathematik, Universität Ulm
e-mail: irene.bouw@uni-ulm.de, e-mail: stefan.wewers@uni-ulm.de

The exponent f_p is called the *conductor exponent* of Y at p . It is known that f_p only depends on the ramification of the local Galois representation associated with Y . In particular, if Y has good reduction at p then $f_p = 0$. If Y has bad reduction at p then the computation of f_p can be quite difficult. Until recently, an effective method for computing f_p was only known for elliptic curves ([24], §IV.10) and for genus 2 curves if $p \neq 2$ ([11]).

It was shown in [2] that f_p can effectively be computed from the *stable reduction* of Y at p . Moreover, for certain families of curves (the *superelliptic curves*) we gave a rather simple recipe for computing the stable reduction. The latter result needed the assumption that p does not divide the degree n . In [18] this restriction is removed for superelliptic curves of prime degree.

In the present paper we systematically study the case of *Picard curves*. These are superelliptic curves of genus 3 and degree 3, given by an equation of the form

$$Y : y^3 = f(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$$

with $f \in \mathbb{Q}[x]$ separable. Picard curves form in some sense the next family of curves to study after hyperelliptic curves. They are interesting for many reasons and have been intensively studied, see e.g. [14], [8], [9], and [16].

Our main results classify all possible configurations for the stable reduction of a Picard curve at a prime p , and use this to determine restrictions on the conductor exponents. For instance, we prove the following.

Theorem 1.1. *Let Y be a Picard curve over \mathbb{Q} .*

- (a) *Then Y has bad reduction at $p = 3$, and $f_3 \geq 4$.*
- (b) *For $p = 2$ we have $f_2 \neq 1$.*
- (c) *For $p \geq 5$ we have $f_p \in \{0, 2, 4, 6\}$.*

Theorem 3.6 is a somewhat stronger version of the first statement. Theorem 4.4 contains the last two statements. We also give explicit examples, showing that at least part of our results are sharp. Our result can be seen as a complement, for Picard curves, to a result of Brumer–Kramer ([3], Theorem 6.2), who prove an upper bound for f_p for abelian varieties of fixed dimension. Since the conductor of a curve coincides with that of its Jacobian, the result applies to our situation, as well. A more careful case-by-case analysis, combined with ideas from [3], could probably be used to obtain a more precise list of possible values for the conductor exponent at $p = 2, 3$, as well.

In the last section we discuss the problem of constructing Picard curves with small conductor. As a consequence of the Shafarevich conjecture (aka Faltings’ Theorem), there are at most a finite number of nonisomorphic curves of given genus and of bounded conductor. But except in very special cases, no effective proof of this theorem is known.

In his recent PhD thesis, the first named author has made an extensive search for Picard curves with good reduction outside a small set of small primes, and computed their conductor. The Picard curve with the smallest conductor that was found is the curve

$$Y : y^3 = x^4 - 1,$$

which has conductor

$$N_Y = 2^6 3^6 = 46656.$$

We propose as a subject for further research to either prove that the above example is the Picard curve over \mathbb{Q} with the smallest possible conductor, or to find (one or all) counterexamples. We believe that the methods presented in this paper may be very helpful to achieve this goal.

2 Semistable reduction

We first introduce the general setup concerning the stable reduction and the conductor exponents of Picard curves. As explained in the introduction, the conductor exponent is a local invariant, encoding information about the ramification of the local Galois representation associated with the curve. Therefore, we may replace the number field K by its strict henselization. In other words, we may work from the start over a henselian field of mixed characteristic with algebraically closed residue field.

2.1 Setup and notation

Throughout Section 2 - 4 the letter K will denote a field of characteristic zero that is henselian with respect to a discrete valuation. We denote the valuation ring by \mathcal{O}_K , the maximal ideal of \mathcal{O}_K by \mathfrak{p} and the residue field by $k = \mathcal{O}_K/\mathfrak{p}$. We assume that k is algebraically closed of characteristic $p > 0$. The most important example for us is when $K = \mathbb{Q}_p^{\text{nr}}$ is the maximally unramified extension of the p -adic numbers. Then $\mathfrak{p} = (p)$ and $k = \bar{\mathbb{F}}_p$.

Let Y/K be a Picard curve, given by the equation

$$Y : y^3 = f(x), \tag{1}$$

where $f \in K[x]$ is a separable polynomial of degree 4. We set $X := \mathbb{P}_K^1$ and interpret (1) as a finite cover $\phi : Y \rightarrow X$, $(x, y) \mapsto x$, of degree 3.

By the Semistable Reduction Theorem (see [5]), there exists a finite extension L/K such that the curve $Y_L := Y \otimes_K L$ has semistable reduction. Since $g(Y) = 3 \geq 2$, there even exists a (unique) distinguished semistable model $\mathcal{Y} \rightarrow \text{Spec } \mathcal{O}_L$ of Y_L , the *stable model* ([5], Corollary 2.7). The special fiber $\bar{Y} := \mathcal{Y}_s$ of \mathcal{Y} is called the *stable reduction* of Y . It is a stable curve over k ([5], § 1), and it only depends on Y , up to unique isomorphism.

It is no restriction to assume that the extension L/K is Galois and contains a third root of unity $\zeta_3 \in L$. Then the cover $\phi_L : Y_L \rightarrow X_L$ (the base change of ϕ to L) is a

Galois cover. Its Galois group G is cyclic of order 3, generated by the element σ which is determined by

$$\sigma(y) = \zeta_3 y.$$

Let $\Gamma := \text{Gal}(L/K)$ denote the Galois group of the extension L/K . The group Γ acts faithfully and in a natural way on the scheme $Y_L = Y \otimes_K L$. We denote by \tilde{G} the subgroup of $\text{Aut}(Y_L)$ generated by G and the image of Γ . By definition, \tilde{G} is a semidirect product,

$$\tilde{G} = G \rtimes \Gamma.$$

The action of Γ on G via conjugation is determined by the following formula: for τ in Γ we have

$$\tau \sigma \tau^{-1} = \begin{cases} \sigma & \text{if } \tau(\zeta_3) = \zeta_3, \\ \sigma^2 & \text{if } \tau(\zeta_3) = \zeta_3^2. \end{cases} \quad (2)$$

Because of the uniqueness properties of the stable model, the action of \tilde{G} on Y_L extends to an action on \mathcal{Y} . By restriction, we see that \tilde{G} has a natural, k -linear action on \bar{Y} . This action will play a decisive role in our analysis of the stable reduction \bar{Y} . For the rest of this subsection we focus on the action of the subgroup $G \subset \tilde{G}$. The role of the subgroup $\Gamma \subset \tilde{G}$ will become important later.

Remark 2.1. (a) The quotient scheme $\mathcal{X} := \mathcal{Y}/G$ is a semistable model of $X_L = \mathbb{P}_L^1$, see e.g. [17], Cor. 1.3.3.i. Since the map $\mathcal{Y} \rightarrow \mathcal{X}$ is finite and \mathcal{Y} is normal, \mathcal{Y} is the normalization of \mathcal{X} in the function field of Y_L . This means that \mathcal{Y} is uniquely determined by a suitable semistable model \mathcal{X} of X_L .

(b) Let $\bar{X} := \mathcal{X} \otimes k$ denote the special fiber of \mathcal{X} and $\bar{\phi} : \bar{Y} \rightarrow \bar{X}$ the induced map. We note that $\bar{\phi}$ is a finite G -invariant map. It is not true in general that $\bar{Y}/G = \bar{X}$. However, the natural map $\bar{Y}/G \rightarrow \bar{X}$ is radicial and in particular a homeomorphism (see e.g. [17], Prop. 2.4.11).

(c) Every irreducible component $W \subset \bar{Y}$ is smooth. To see this note that the quotient of W by its stabilizer in G is homeomorphic to an irreducible component $Z \subset \bar{X}$, which is a smooth curve of genus 0. If W has a singular point, then σ acts on W and permutes the two branches of W passing through this point. But since σ has order 3, this is impossible.

Let $\Delta_{\bar{Y}}$ denote the component graph of \bar{Y} : the vertices are the irreducible components of \bar{Y} and the edges correspond to the singular points. The stability condition for \bar{Y} means that an irreducible component of genus 0 corresponds to a vertex of $\Delta_{\bar{Y}}$ of degree ≥ 3 . The number of loops of $\Delta_{\bar{Y}}$ is given by the well known formula

$$\gamma(\bar{Y}) := \dim_{\mathbb{Q}} H^1(\Delta_{\bar{Y}}, \mathbb{Q}) = r - s + 1, \quad (3)$$

where r is the number of edges and s the number vertices of $\Delta_{\bar{Y}}$.

The curve \bar{X} is also semistable, but in general not stable. Since \bar{X} has arithmetic genus 0, the component graph $\Delta_{\bar{X}}$ is a tree, and every vertex corresponds to a smooth curve of genus 0. It follows from Remark 2.1 that $\Delta_{\bar{X}} = \Delta_{\bar{Y}}/G$.

Lemma 2.2. *If $W \subset \bar{Y}$ is an irreducible component, then $\sigma(W) = W$.*

Proof. To derive a contradiction, we assume that $W_1, W_2, W_3 \subset \bar{Y}$ are three distinct components that form a single G -orbit. Then $W_i \xrightarrow{\sim} Z := \bar{\phi}(W_i)$. Since Z is a component of \bar{X} , we conclude that $g(W_i) = 0$, for $i = 1, 2, 3$. The stability condition on \bar{Y} implies that each W_i contains at least three singular points of \bar{Y} . Hence Z also contains at least three singular points of \bar{X} .

Let $\bar{Y} \rightarrow \bar{Y}_0$ denote the unique morphism which contracts all components of \bar{Y} except the W_i and which is an isomorphism on the intersection of $\cup_i W_i$ with the smooth locus of \bar{Y} . Similarly, let $\bar{X} \rightarrow \bar{X}_0$ be the map contracting all components of \bar{X} except Z . These maps fit into a commutative diagram

$$\begin{array}{ccc} \bar{Y} & \longrightarrow & \bar{Y}_0 \\ \downarrow & & \downarrow \\ \bar{X} & \longrightarrow & \bar{X}_0, \end{array}$$

where the vertical arrows are quotient maps by the group G (at least for the underlying topological spaces). Also, $\bar{X}_0 \cong Z$.

Let $\bar{x} \in Z$ be one of the singular point of \bar{X} lying on Z , and let $T \subset \bar{X}$ the closed subset which is contracted to $\bar{x} \in Z = \bar{X}_0$. Then T is a nonempty and connected union of irreducible components of \bar{X} and hence a semistable curve of genus 0. In particular, the component graph of T is a tree. Let $Z' \subset T$ be a tail component. As a component of \bar{X} , Z' intersects the rest of \bar{X} in at most two points. Let $W' \subset \bar{Y}$ be an irreducible component lying above Z' . The stability of \bar{Y} implies that $\sigma(W') = W'$ and that the action of σ on W' is nontrivial. (Otherwise W' would be homeomorphic to Z' , and hence W' would be a component of genus 0 intersecting the rest of \bar{Y} in at most two points.) It follows that the inverse image $S \subset \bar{Y}$ of T is connected. Note that S meets the component W_i in the unique point on W_i above \bar{x} . Since S is connected, it follows that the map $\bar{Y} \rightarrow \bar{Y}_0$ contracts S to a single point.

We conclude that the curve \bar{Y}_0 has at least three distinct singular points where all three components W_i meet. Equation (3) implies that $\gamma(\bar{Y}_0)$ is at least 1. It follows that the arithmetic genus of \bar{Y}_0 is ≥ 4 , and hence $g(\bar{Y}) \geq 4$ as well. This is a contradiction, and the lemma follows. \square

2.2 The conductor exponent

Let $\mathfrak{c}_{\mathfrak{p}}$ be the *conductor* of the $\text{Gal}(\bar{K}/K)$ -representation $H_{\text{et}}^1(Y_{\bar{K}}, \mathbb{Q}_{\ell})$, see [22]. By definition, this is an ideal of \mathcal{O}_K of the form

$$\mathfrak{c}_{\mathfrak{p}} = \mathfrak{p}^{f_{\mathfrak{p}}},$$

with $f_{\mathfrak{p}} \geq 0$. The integer $f_{\mathfrak{p}}$ is called the *conductor exponent* of Y/K .¹

¹ When working in a local context, $f_{\mathfrak{p}}$ is often simply called the conductor of Y .

We recall from [2] an explicit formula for f_p , in terms of the action of $\Gamma = \text{Gal}(L/K)$ on \bar{Y} . For this we let $\Gamma^u \subset \Gamma$, for $u \geq 0$, denote the u th higher ramification group (in the upper numbering). We set $\bar{Y}^u := \bar{Y}/\Gamma^u$. Note that \bar{Y}^u is a semistable curve for all u . Note also that $\Gamma = \Gamma^0$ because the residue field k is assumed to be algebraically closed.

Proposition 2.3. *The conductor exponent of the curve Y/K is given by*

$$f_p = \varepsilon + \delta, \quad (4)$$

where

$$\varepsilon := 6 - \dim H_{\text{et}}^1(\bar{Y}^0, \mathbb{Q}_\ell) \quad (5)$$

and

$$\delta := \int_0^\infty (6 - 2g(\bar{Y}^u)) du. \quad (6)$$

Proof. See [2], Theorem 2.9 and [1], Corollary 2.14. \square

The étale cohomology group $H_{\text{et}}^1(\bar{Y}^u, \mathbb{Q}_\ell)$ decomposes as

$$H_{\text{et}}^1(\bar{Y}^u, \mathbb{Q}_\ell) = \bigoplus_W H_{\text{et}}^1(W, \mathbb{Q}_\ell) \oplus H^1(\Delta_{\bar{Y}^u}, \mathbb{Q}_\ell),$$

where the first sum runs over the set of irreducible components W of the normalization of \bar{Y}^u and $\Delta_{\bar{Y}^u}$ is the graph of components of \bar{Y}^u . (See [2], Lemma 2.7.(1).) Therefore, the second term in (5) can be written as

$$\dim H_{\text{et}}^1(\bar{Y}^0, \mathbb{Q}_\ell) = \sum_W \dim H_{\text{et}}^1(W, \mathbb{Q}_\ell) + \dim H^1(\Delta_{\bar{Y}^0}). \quad (7)$$

The arithmetic genus of \bar{Y}^u , which occurs in (6), is given by the formula

$$g(\bar{Y}^u) = \sum_W g(W) + \dim H^1(\Delta_{\bar{Y}^u}). \quad (8)$$

For future reference we note that $\dim H_{\text{et}}^1(W, \mathbb{Q}_\ell) = 2g(W)$. The integer $\gamma(\bar{Y}^0) := \dim H^1(\Delta_{\bar{Y}^0})$ can be interpreted as the number of loops of the graph $\Delta_{\bar{Y}^0}$. It is bounded by $g(\bar{Y}^0)$, and hence by $g(Y) = 3$.

Lemma 2.4. *The following statements are equivalent.*

- (a) $\delta = 0$.
- (b) Γ^u acts trivially on \bar{Y} , for all $u > 0$.
- (c) The curve Y has semistable reduction over a tamely ramified extension of K .

Proof. Assume that $\delta = 0$. By (6) this means that $3 = g(\bar{Y}) = g(\bar{Y}^u)$ for all $u > 0$. Using (8) one easily shows that this means that Γ^u acts trivially on the component graph $\Delta_{\bar{Y}}$ of \bar{Y} . Moreover, for every component $W \subset \bar{Y}$ we have $g(W) = g(W/\Gamma^u)$. It follows that Γ^u acts trivially on \bar{Y} . We have proved the implication (a) \Rightarrow (b). The implication (b) \Rightarrow (c) follows from [12], Theorem 4.44. The implication (c) \Rightarrow (a) follows immediately from the definition of δ . \square

3 The wild case: $p = 3$

In this section we assume that $p = 3$. We first analyze the special fiber of the stable model of Y_L , and show that there are essentially five reduction types. From § 3.2 we consider the case where K is absolutely unramified, and derive a lower bound for the conductor exponent f_3 .

3.1 The stable model

We keep all the notation introduced in § 2. In addition, we assume that $p = 3$. Lemma 2.2 implies that we can distinguish between two types of irreducible components of \bar{Y} .

Definition 3.1. An irreducible component $W \subset \bar{Y}$ is called *étale* if the restriction $\sigma|_W \in \text{Aut}_k(W)$ is nontrivial. If $\sigma|_W$ is the identity, then W is called an *inseparable component*.

Let $W \subset \bar{Y}$ be an irreducible component, and let $Z := \bar{\phi}(W) \subset \bar{X}$ be its image. Then Z is an irreducible component of \bar{X} and hence a smooth curve of genus 0. Lemma 2.2 shows that $\sigma(W) = W$. It follows that $W/G \rightarrow Z$ is a homeomorphism. If W is an inseparable component, then $W \rightarrow Z$ is a purely inseparable homeomorphism (since $W \rightarrow Z$ has degree 3, this can only happen when $p = 3$). It follows that every inseparable component has genus zero.

If W is an étale component, then $Z \cong W/G$, and $W \rightarrow Z$ is a G -Galois cover. For future reference we recall that the Riemann–Hurwitz formula for wildly ramified Galois covers of curves yields

$$2g(W) - 2 = -2 \cdot 3 + \sum_z 2(h_z + 1), \quad (9)$$

where the sum runs over the branch points of $W \rightarrow Z$ and h_z is the (unique) jump in the filtration of the higher ramification groups in the lower numbering. We have that $h_z \geq 1$ is prime to p ([21], § IV.2, Cor. 2 to Prop. 9).

Theorem 3.2. *We are in exactly one of the following five cases.*

- (a) *The curve \bar{Y} is smooth and irreducible.*
- (b) *There are exactly two components W_1, W_2 which are both étale, meet in a single point, and have genus $g(W_1) = 2$, $g(W_2) = 1$.*
- (c) *There are three étale components W_1, W_2, W_3 of genus one, and one inseparable component W_0 of genus zero. For $i = 1, 2, 3$, W_i intersects W_0 in a unique point, and these intersection points are precisely the singular points of \bar{Y} .*
- (d) *There are two components W_1, W_2 which are étale of genus $g(W_1) = 1$, $g(W_2) = 0$. There are exactly three singular points, which form an orbit under the action of G , and where W_1 and W_2 meet.*

(e) *There are three components W_1, W_2, W_3 , which are étale and of genus $g(W_1) = g(W_2) = 0$ and $g(W_3) = 1$, and four singular points. Three of the singular points are points of intersection of W_1 and W_2 , and form an orbit under the action of G . The fourth singular point is the point of intersection of W_2 and W_3 .*

Proof. Let r_1 (resp. s_1) be the number of singular points (resp. irreducible components) of \bar{Y} which are fixed by σ , and let r_2 (resp. s_2) be the number of orbits of singular point (resp. irreducible components) of \bar{Y} of length 3. Lemma 2.2 states that $s_2 = 0$. Therefore, (3) becomes

$$\gamma(\bar{Y}) = r - s + 1 = r_1 + 3r_2 - s + 1. \quad (10)$$

Because $\Delta_{\bar{X}} = \Delta_{\bar{Y}}/G$ is a tree, we have

$$\gamma(\bar{X}) := \dim H^1(\Delta_{\bar{X}}) = r_1 + r_2 - s + 1 = 0. \quad (11)$$

Combining (10) and (11) we obtain

$$\gamma(\bar{Y}) = 2r_2. \quad (12)$$

Since $0 \leq \gamma(\bar{Y}) \leq 3$, we conclude that $\gamma(\bar{Y}) \in \{0, 2\}$ and $r_2 \in \{0, 1\}$.

Case 1: $r_2 = 0$ and $\gamma(\bar{Y}) = 0$.

In this case $\Delta_{\bar{Y}}$ is a tree, and the sum of the genera of all irreducible components is 3. In particular, there are at most 3 components of genus > 0 . Moreover, the stability condition implies that every component of genus zero contains at least three singular points of \bar{Y} . It is an easy combinatorial exercise to see that this leaves us with exactly four possibilities for the tree $\Delta_{\bar{Y}}$. Going through these four cases we will see that one of them is excluded, while the remaining three correspond to Case (a), (b), and (c) of Theorem 3.2.

The first case is when \bar{Y} has a unique irreducible component. Then \bar{Y} is smooth. This is Case (a) of the lemma. Secondly, there may be two irreducible components, of genus 1 and 2, and a unique singular point. This corresponds to Case (b).

Thirdly, there may be three irreducible components, each of genus 1, and two singular points. We claim that this case cannot occur. Indeed, one of the three components would contain two singular points, and each of these two points must be a fixed point of σ . It follows that the G -cover $W \rightarrow Z = W/G$ is ramified in at least two points. The Riemann–Hurwitz formula (9) implies that $g(W) \geq 2$. This yields a contradiction, and we conclude that this case does not occur.

Finally, in the last case, there are four singular points and four irreducible components. Three of them have genus 1 and one has genus zero. The component of genus zero necessarily contains all three singular points. A similar argument as in the previous case shows that the genus-0 component cannot be étale. This corresponds to Case (c).

Case 2: $\gamma(\bar{Y}) = 2$ and $r_2 = 1$.

In this case the sum of the genera of all components is equal to 1. Therefore, there

must be a unique component of genus 1, and all other components have genus 0. Let W_1 and W_2 be two components which meet in a singular point \bar{y} such that $\sigma(\bar{y}) \neq \bar{y}$. Since $\sigma(W_i) = W_i$ for $i = 1, 2$ (Lemma 2.2), W_1 and W_2 are étale components and intersect each other in exactly three points (the G -orbit of \bar{y}).

If there are no further components, we are in Case (d). Assume that there exists a third component W_3 . Let $T \subset \bar{Y}$ be the maximal connected union of components which contains W_3 but neither W_1 nor W_2 . Then T contains a unique component W_0 which meets either W_1 or W_2 in a singular point. The component graph of T is a tree, and we consider W_0 as its root. By the stability condition, every tail component of T must have positive genus, so T has a unique tail. If W_0 is not this tail, it has genus 0 and intersects the rest of \bar{Y} in exactly 2 points. This contradicts the stability condition. We conclude that \bar{Y} has exactly three components, of genus $g(W_1) = g(W_2) = 0$ and $g(W_3) = 1$. This is Case (e) of the lemma. Now the proof is complete. \square

3.2 A lower bound for f_3

We continue with the assumptions from the previous subsection. In addition, we assume that K is absolutely unramified. By this we mean that $\mathfrak{p} = (3)$. Under this assumption, we prove a lower bound for the conductor exponent $f_3 := f_{\mathfrak{p}}$. In fact, we will give a lower bound for ε , where $f_3 = \varepsilon + \delta$ is the decomposition from Proposition 2.3. If L/K is at most tamely ramified, then $\delta = 0$ (Lemma 2.4). In this case, our bounds are sharp.

Since K is absolutely unramified, the third root of unity $\zeta_3 \in L$ is *not* contained in K . Therefore, there exists an element $\tau \in \Gamma = \text{Gal}(L/K)$ such that $\tau(\zeta_3) = \zeta_3^2$. Let m be the order of τ . After replacing τ by a suitable odd power of itself we may assume that m is a power of 2. We keep this notation fixed for the rest of this paper. Recall that the semidirect product $\tilde{G} = G \rtimes \Gamma$ acts on \bar{Y} in a natural way.

The following observation is crucial for our analysis of the conductor exponent.

Lemma 3.3. *Let $W \subset \bar{Y}$ be an étale component such that $\tau(W) = W$. Then inside the automorphism group of W we have*

$$\tau \circ \sigma \circ \tau^{-1} = \sigma^2 \neq \sigma. \quad (13)$$

In particular, $\tau|_W$ is nontrivial.

Proof. The statement follows immediately from Equation (2) and Definition 3.1. \square

Despite its simplicity, Lemma 3.3 has the following striking consequence. Note that we consider potentially good but not good reduction as bad reduction in this paper.

Proposition 3.4. *Assume that $\mathfrak{p} = (3)$. Then every Picard curve Y over K has bad reduction.*

Proof. Lemma 3.3 implies that Y acquires semistable reduction only after passing to a ramified extension $L \ni \zeta_3$. Therefore Y/K does not have good reduction. The fact that $f_3 \neq 0$ follows from Proposition 2.3, together with the fact that τ acts nontrivially on each irreducible component of \bar{Y} (Lemma 3.3). \square

In order to prove more precise lower bounds for f_3 , we need to analyze the action of σ and τ on \bar{Y} in more detail.

Lemma 3.5. *Let $W \subset \bar{Y}$ be an étale component. Then one of the following cases occurs:*

$g(W)$	r	h	$g(W/\Gamma^0)$
0	1	1	0
1	1	2	0
2	2	(1, 1)	1
3	1	4	0

Here r is the number of ramification points of the G -cover $W \rightarrow Z := W/G$ and h lists the set of lower jumps. The fourth column gives an upper bound for the genus of W/Γ^0 .

Proof. Recall that we have assumed that the order m of τ is a power of 2.

The Riemann–Hurwitz formula (9) immediately yields the cases for $g(W)$, r , and h stated in the lemma, together with one additional possibility: the curve W has genus 3 and $\phi : W \rightarrow Z \cong \mathbb{P}^1$ is branched at two points, with lower jump 1 and 2, respectively. We claim that this case does not occur.

Assume that W is an étale component of \bar{Y} such that $\phi : W \rightarrow Z$ is branched at 2 points. Lemma 3.3 implies that τ acts nontrivially on W . Since τ normalizes σ and the two ramification points have different lower jumps, it follows that τ fixes both ramification points w_i of ϕ . We conclude that $H := \langle \sigma, \tau \rangle$ acts on W as a nonabelian group of order 6 fixing the v_i .

We write h_i for lower jump of w_i . Lemma 2.6 of [15] implies that $\gcd(h_i, m)$ is the order of the prime-to-3 part of the centralizer of H . Since $\gcd(h_1, m) \neq \gcd(h_2, m)$ we obtain a contradiction, and conclude that this case does not occur.

We compute an upper bound for the genus of $W/\langle \tau \rangle$ in each of the remaining cases. This is also an upper bound for $g(W/\Gamma^0)$.

In the case that $g(W) = 0$ there is nothing to prove. In the case that $g(W) = 1$, the automorphism τ fixes the unique ramification point of ϕ , hence $g(W/\Gamma^0) = 0$.

Assume that $g(W) = 2$. The Riemann–Hurwitz formula immediately implies that that $g(W/\langle \tau \rangle) \leq 1$.

Finally, we consider the case that $g(W) = 3$, i.e. Y has potentially good reduction. As before, we have that τ fixes the unique fixed point of σ . Put $H = \langle \sigma, \tau \rangle$. Lemma 3.3 together with the assumption that the order m of τ is a power of 2 implies that the order of the prime-to- p centralizer of H is $\gcd(h = 4, m) = m/2$. It follows that $m = 8$. Since τ has at least one fixed point on W , namely the point at ∞ , the Riemann–Hurwitz formula implies that $g(W/\langle \tau \rangle) = 0$. This finishes the proof of the lemma. \square

We have now all the necessary tools to prove our main theorem.

Theorem 3.6. *Assume $p = (3)$, and let Y be a Picard curve over K . The conductor exponent f_3 of Y/K satisfies*

$$f_3 \geq 4.$$

Moreover:

- (a) *If $f_3 \leq 6$ then Y achieves semistable reduction over a tamely ramified extension L/K .*
- (b) *If $f_3 = 4$ then we are in Case (b) or Case (c) from Theorem 3.2.*
- (c) *If $f_3 = 5$ then we are in Case (d) or in Case (e) of Theorem 3.2.*

Proof. We use the assumptions and notations from the beginning of § 3.2. Recall that the inertia subgroup $\Gamma^0 \subset \Gamma := \text{Gal}(L/K)$ acts on the geometric special fiber \bar{Y} of the stable model of Y_L and that the quotient $\bar{Y}^0 = \bar{Y}/\Gamma^0$ is again a semistable curve.

Claim: We have that

$$\dim H_{\text{et}}^1(\bar{Y}^0, \mathbb{Q}_\ell) \leq 2. \quad (14)$$

Note that (14), together with (4) and (5), immediately implies the first statement $f_3 \geq 4$ of the theorem.

Recall from (8) and (3) that the contribution of a smooth component W of \bar{Y}^0 to $\dim H_{\text{et}}^1(\bar{Y}^0, \mathbb{Q}_\ell)$ is $2g(W)$. The contribution of $H^1(\Delta_{\bar{Y}})$ to $\dim H_{\text{et}}^1(\bar{Y}^0, \mathbb{Q}_\ell)$ is $\gamma(\bar{Y}^0)$, which is less than or equal to $g(\bar{Y}^0)$.

Let $W \subset \bar{Y}$ be an irreducible component, and denote by $W^0 \subset \bar{Y}_0$ its image in \bar{Y}^0 . Clearly, $g(W^0) \leq g(W)$. Moreover, if $\tau(W) = W$ then Lemma 3.5 shows that $g(W^0) \leq 1$.

Let us consider each case of Theorem 3.2 separately. In Case (a), \bar{Y} is smooth and irreducible of genus 3. Then \bar{Y}^0 is also smooth and irreducible, and Lemma 3.5 shows that $g(\bar{Y}^0) = 0$. So in Case (a) we have proved $\dim H_{\text{et}}^1(\bar{Y}^0, \mathbb{Q}_\ell) = 0$, which is strictly stronger than (14). Similarly, in Case (b) Lemma 3.5 shows that \bar{Y}^0 consists of two irreducible components which meet in a single point. One of these components has genus zero, the other one has genus ≤ 1 . Therefore, (14) holds in Case (b).

Assume that we are in Case (c). Let W_1, W_2, W_3 denote the three components of genus 1, and $W_i^0, i = 1, 2, 3$, their images in \bar{Y}^0 . Since the order of τ is a power of two, τ fixes exactly one of these components (say W_1), or all three. In the first case, $g(W_1^0) = 0$ by Lemma 3.5, and $W_2^0 = W_3^0$. Therefore, $\dim H_{\text{et}}^1(\bar{Y}^0, \mathbb{Q}_\ell) = 1$. In the second case, $g(W_i^0) = 0$ for $i = 1, 2, 3$, and $\dim H_{\text{et}}^1(\bar{Y}^0, \mathbb{Q}_\ell) = 0$. In both cases, (14) holds.

Now assume that we are in Case (d). The action of Γ^0 must fix both components W_1, W_2 , since $g(W_1) \neq g(W_2)$. Lemma 3.5 shows that $g(W_i/\Gamma^0) = 0$, for $i = 1, 2$. Also, τ permutes the three singular points of \bar{Y} . But these points form one orbit under the action of G . Hence it follows from (13) that τ fixes exactly one singular point and permutes the other two. We conclude that the curve \bar{Y}^0 has two smooth components

of genus 0 which meet in at most two points. We conclude that $\dim H_{\text{et}}^1(\bar{Y}^0, \mathbb{Q}_\ell) \leq 1$. A similar analysis shows that the same conclusion holds in Case (e). This proves the claim (14).

While proving the claim, we have shown the following stronger conclusion:

$$\dim H_{\text{et}}^1(\bar{Y}^0, \mathbb{Q}_\ell) \in \begin{cases} \{0\}, & \text{Case (a),} \\ \{0, 2\}, & \text{Case (b), (c),} \\ \{0, 1\}, & \text{Case (d), (e).} \end{cases} \quad (15)$$

It follows that $\varepsilon = 6$ in Case (a), $\varepsilon \in \{4, 6\}$ in the Cases (b) and (c), and $\varepsilon \in \{5, 6\}$ in the Cases (d) and (e).

The remaining statement that Y acquires stable reduction over a tamely ramified extension L of K in the case that $f_3 \leq 6$ follows from Lemma 2.4. \square

Corollary 3.7. *If $\mathfrak{p} = (3)$ and Y has potentially good reduction, then $f_3 \geq 6$.*

3.3 Examples

In this section we discuss two explicit examples of Picard curves over \mathbb{Q}_3^{nr} in some detail. These examples show, among other things, that the lower bounds for f_3 given by Theorem 3.6 are sharp.

Let us fix some notation. We set $K := \mathbb{Q}_3^{\text{nr}}$. Given a suitable finite extension L/K , we denote by v_L the unique extension of the 3-adic valuation to L (which is normalized such that $v_L(3) = 1$). We let $F(X_L)$ denote the function field of $X_L := \mathbb{P}_L^1$, and identify $F(X_L)$ with the rational function field $L(x)$. For a Picard curve Y over K given by $y^3 = f(x)$ for a quartic polynomial $f \in K[x]$ the function field $F(Y_L)$ of Y_L is the degree-3 extension of $F(X_L)$ obtained by adjoining the function y .

Let \mathcal{X} be a semistable model of X_L , and let $Z_1, \dots, Z_n \subset \bar{X} := \mathcal{X} \otimes \mathbb{F}_L$ denote the irreducible components of the special fiber. Since each Z_i is a prime divisor on \mathcal{X} , it gives rise to a discrete valuation v_i on $F(X_L)$, extending v_L . It has the property that the residue field of v_i can be naturally identified with the function field of Z_i . Since X_L is simply a projective line and \mathcal{X} is a semistable model, the valuations v_i have a simple description, as follows. For all i , there exists a coordinate $x_i \in F(X_L)$ such that v_i is the Gauss valuation on $F(X_L) = L(x_i)$ with respect to x_i . The coordinate x_i is related to x by a fractional linear transformation

$$x = \frac{a_i x_i + b_i}{c_i x_i + d_i},$$

with $a_i d_i - b_i c_i \neq 0$. It can be shown that the model \mathcal{X} is uniquely determined by the set $\{v_1, \dots, v_n\}$, see [2] or [19].

Let \mathcal{Y} denote the normalization of \mathcal{X} inside the function field $F(Y_L)$. Then \mathcal{Y} is a normal integral model of Y_L . In general, \mathcal{Y} has no reason to be semistable, and it is not clear in general how to describe its special fiber $\bar{Y} := \mathcal{Y} \otimes k$. However, each

irreducible component $W \subset \bar{Y}$ corresponds again to a discrete valuation w on $F(Y_L)$ extending v_L , such that the residue field of w is the function field of W . It can be shown that this gives a bijection between the irreducible components of \bar{Y} and the set of discrete valuations on $F(Y_L)$ extending one of the valuations v_i (see e.g. [19], § 3). In many situations, the knowledge of all extensions of the v_i to $F(Y_L)$ will give enough information to decide whether the model \mathcal{Y} is semistable and to describe its special fiber.

We need one more piece of notation. For $m > 1$ prime to 3 we set

$$L_m := K(\pi)/K$$

where $\pi^m = -3$. Then L_m/K is a tamely ramified Galois extension of degree m . The Galois group $\Gamma := \text{Gal}(L_m/K)$ is cyclic and generated by the element $\tau \in \Gamma_m := \text{Gal}(L_m/K)$ determined by

$$\tau(\pi) = \zeta_m \pi,$$

where $\zeta_m \in K$ is a primitive m th root of unity (which exists because k is algebraically closed). Note also that L_m contains the third root of unity

$$\zeta_3 := \frac{-1 + \pi^{m/2}}{2}.$$

We remark that the choice of τ and m agrees with the notation chosen in § 3.2

Example 3.8. Let Y be the Picard curve over K given by the equation

$$y^3 = x^4 + 1. \tag{16}$$

We claim that Y has potentially good reduction, which is attained over the tame extension $L := L_8 = K(\pi)/K$, with $\pi^8 = -3$.

To prove this, we apply the coordinate changes

$$x = \pi^3 x_1, \quad y = 1 + \pi^4 y_1$$

to (16). After a brief calculation, we obtain the new equation

$$y_1^3 - \pi^4 y_1^2 - y_1 = x_1^4. \tag{17}$$

Equation (17) is equivalent to (16) in the sense that it defines a curve over K which is isomorphic to Y . Also, (17) defines an integral model \mathcal{Y} of Y_L . Its special fiber is the curve over $k = \bar{\mathbb{F}}_3$ given by the (affine) equation

$$\bar{Y} : y_1^3 - y_1 = x_1^4.$$

This is a smooth curve of genus 3. It follows that \mathcal{Y} has good reduction over L , as claimed.

Since Y acquires stable reduction over a tame extension L/K , Lemma 2.4 implies that $f_3 = \varepsilon$. Equations (5) and (15) imply that $f_3 = 6$.

For completeness, we compute the action of $\Gamma^0 = \langle \tau \rangle$ on \bar{Y} explicitly. We consider τ as an automorphism of the structure sheaf of \mathcal{Y} . By definition, we have

$$\tau(\pi) = \zeta_8 \pi, \quad \tau(x) = x, \quad \tau(y) = y.$$

It follows that

$$\tau(x_1) = \zeta_8^5 x_1, \quad \tau(y_1) = -y_1.$$

This describes $\tau|_{\bar{Y}}$ as an automorphism of \bar{Y} of order 8, as expected from the proof of Lemma 3.5.

Example 3.9. Let Y/K be the Picard curve

$$Y : y^3 = f(x) := 3x^4 + x^3 - 54. \quad (18)$$

We claim that Y has semistable reduction over the tame extension $L := L_4/K$. Moreover, the stable reduction \bar{Y} is as in Case (b) of Theorem 3.2, and $f_3 = 4$.

First we define a semistable model \mathcal{X} of $X_L := \mathbb{P}_L^1$ by specifying two discrete valuations v_1, v_2 on $F(X_L)$ which extend v_L . We then show that the normalization \mathcal{Y} of \mathcal{X} in $F(Y_L)$ is the stable model of Y_L , and determine its special fiber \bar{Y} and the action of the inertia group of L/K on \bar{Y} .

The valuation v_1 is defined as the Gauss valuation on $F(X_L) = F(x_1)$ with respect to the coordinate x_1 , which is related to x by

$$x = \pi^2 x_1. \quad (19)$$

We claim that v_1 has a unique extension w_1 to $F(Y_L)$ that is unramified. To show this, we need a so-called *p-approximation* of f with respect to v_1 , see [18]. In fact, we can write

$$f = \pi^6 (x_1^3 + \pi^6 (2 - x_1^4)).$$

Here we have used the relation $\pi^4 = -3$. This suggests the coordinate change

$$y = \pi^2 (x_1 + \pi^2 y_1). \quad (20)$$

After a short calculation we obtain a new equation for Y_L :

$$y_1^3 - \pi^2 x_1 y_1^2 - x_1^2 y_1 = 2 - x_1^4. \quad (21)$$

If we consider (21) as defining an affine curve over \mathcal{O}_L , its special fiber is the affine curve over k with equation

$$\bar{y}_1^3 - \bar{x}_1^2 \bar{y}_1 = -1 - \bar{x}_1^4. \quad (22)$$

In fact, (22) defines an irreducible affine curve with a cusp singularity in $(\bar{x}_1, \bar{y}_1) = (0, -1)$. It follows that the inverse image in \bar{Y} of \bar{X}_1 is an irreducible component W_1 of multiplicity one birationally equivalent to the curve given by (22). To compute the geometric genus of W_1 we substitute $\bar{y}_1 = -1 + \bar{x}_1 \bar{z}_1$ into (22) and obtain the Artin–Schreier equation

$$\bar{z}_1^3 - \bar{z}_1 = -\bar{x}_1^{-1} - \bar{x}_1. \quad (23)$$

Using the Riemann–Hurwitz formula, one sees that W_1 has geometric genus 2.

The valuation v_2 of $F(X_L)$ corresponds to the choice of the coordinate x_2 given by

$$x = 3(1 + \pi x_2). \quad (24)$$

After a short calculation we can write

$$f = 3^3((-1 + \pi x_2)^3 - 2\pi^6 x_2^2 + 3^2(\dots)). \quad (25)$$

This suggests the change of coordinate

$$y = 3((-1 + \pi x_2) + \pi^2 y_2). \quad (26)$$

Plugging in (26) into (18) and using (25) we arrive at the equation

$$y_2^3 + \pi^2(-1 + \pi x_2)y_2^2 - (-1 + \pi x_2)^2 y_2 = -2x_2^2 + \pi^2(\dots). \quad (27)$$

Reducing (27) modulo π we obtain the irreducible equation

$$\bar{y}_2^3 - \bar{y}_2 = \bar{x}_2^2, \quad (28)$$

which defines a curve of genus 1. It follows that the inverse image of \bar{X}_2 in \bar{Y} is an irreducible projective curve W_2 of geometric genus 1.

So \bar{Y} consists of two irreducible components W_1 and W_2 of geometric genus 2 and 1. On the other hand, \mathcal{O}_S is known to have arithmetic genus 3. By a standard argument (see e.g.) we can conclude that W_1, W_2 are smooth and meet transversely in a single point. This shows that Y has semistable reduction over the tame extension L_4/K , with a stable model of type (b).

Let us try to analyze the action of $\Gamma = \Gamma^0 = \langle \tau \rangle$ on \bar{Y} . By definition, $\tau(\pi) = \zeta_4 \pi$, $\tau(x) = x$ and $\tau(y) = y$. From (19) and (20) we deduce that $\tau|_{W_1}$ is given by

$$\tau(\bar{x}_1) = -\bar{x}_1, \quad \tau(\bar{y}_1) = \bar{y}_1, \quad \tau(\bar{z}_1) = -\bar{z}_1.$$

From (24) and (26) we see that

$$\tau(\bar{x}_2) = \zeta_4^3 \bar{x}_2, \quad \tau(\bar{y}_2) = -\bar{y}_2.$$

It follows that the curve $\bar{Y}^0 := \bar{Y}/\Gamma^0$ has two irreducible smooth components, $W_1^0 = W_1/\Gamma^0$ and $W_2^0 = W_2/\Gamma^0$, meeting in a single point. An easy calculation (compare with the proof of Lemma 3.5) shows that $g(W_1^0) = 1$ and $g(W_2^0) = 0$. It follows that $g(\bar{Y}^0) = 1$ and $\dim H^1(\Delta_{\bar{Y}^0}) = 0$ and hence $f_3 = 6 - 2 = 4$.

Remark 3.10. The two examples discussed above are quite special. Typically, the extension L/K needs to be wildly ramified, and have rather large degree. It is then hard (and often practically impossible) to do computations as above by hand. Most of the examples in [4] and this paper have been computed with the help of (earlier

versions of) Julian R  th's Sage packages `mac_lane` and `completion` (available at <https://github.com/saraedum>), and the algorithms from [2] and [18].

4 The tame case: $p \neq 3$

In this section we assume that the residue characteristic p of our ground field K is different from 3. In this case it is much easier to analyze the semistable reduction of Picard curves and to compute the conductor exponent f_p than for $p = 3$. The theoretical background for this are the *admissible covers*, see [7], [12], § 10.4.3, or [26]. In the case of superelliptic curves the computation of f_p has already been described in detail in [2], hence we can be much briefer than in the previous section.

4.1 The stable model

Let K be as in § 2.1, with $p \neq 3$. Let Y/K be a Picard curve, given by an equation

$$Y : y^3 = f(x),$$

where $f \in K[x]$ is a separable polynomial of degree 4. Let L_0/K denote the splitting field of f . Let L/L_0 be a finite extension with ramification index 3 such that L/K is a Galois extension. Then [2], Corollary 4.6 implies that Y acquires semistable reduction over L .

We note in passing that L/K is tamely ramified unless $p = 2$. This follows from the definition of the Galois extension L_0/K , whose degree divides $4! = 24$.

A semistable model \mathcal{Y} of Y_L may be constructed as follows, see [2], § 4. Let $D \subset X = \mathbb{P}_K^1$ denote the branch divisor of the cover $\phi : Y \rightarrow X$, consisting of the set of zeros of f and ∞ . Since L contains the splitting field of f , the pullback $D_L \subset Y_L$ consists of 5 distinct L -rational points. Let $(\mathcal{X}, \mathcal{D})$ denote the *stably marked model* of (X_L, D_L) . By this we mean that \mathcal{X} is the minimal semistable model of X_L with the property that the schematic closure $\mathcal{D} \subset \mathcal{X}$ of D_L is étale over $\text{Spec } \mathcal{O}_L$ and contained inside the smooth locus of $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_L$. Let $\tilde{X} := \mathcal{X} \otimes \mathbb{F}_L$ denote the special fiber of \mathcal{X} and $\tilde{D} = \mathcal{D} \cap \tilde{X}$ the specialization of D_L . Then (\tilde{X}, \tilde{D}) is a stable 5-marked curve of genus zero. This means that \tilde{X} is a tree of projective lines, where every irreducible component has at least three points which are either marked (i.e. lie in the support of \tilde{D}) or are singular points of \tilde{X} .

Let \mathcal{Y} denote the normalization of \mathcal{X} with respect to the cover $Y_L \rightarrow X_L$. Theorem 3.4 from [2] shows that \mathcal{Y} is a quasi-stable model of Y_L . A priori, it is not clear whether \mathcal{Y} is the stable model of Y . The following case-by-case analysis will show that it is.

We will use the fact that the natural map $\mathcal{Y} \rightarrow \mathcal{X}$ is an *admissible cover* with branch locus \mathcal{D} . In particular, the induced map

$$\bar{\phi} : \bar{Y} \rightarrow \bar{X}$$

between the special fiber of \mathcal{Y} and of \mathcal{X} is generically étale and identifies \bar{X} with the quotient scheme \bar{Y}/G .

We describe the restriction of the map $\bar{\phi}$ to an irreducible component \bar{X}_i of \bar{X} . Without loss of generality we may assume that K (and hence L) contains a primitive 3rd root of unity ζ_3 , which we fix. For each branch point ξ of $\bar{\phi}|_{\bar{X}_i}$ the *canonical generator of inertia* $g \in G$ is characterized by $g^*u \equiv \zeta_3 u \pmod{u^2}$, where u is a local parameter at $\bar{\phi}|_{\bar{X}_i}^{-1}(\xi_i)$. A branch point of $\bar{\phi}|_{\bar{X}_i}$ is either the specialization of a branch point of ϕ or a singular point of \bar{X} .

Assume that ξ is the specialization of a branch point. An elementary calculation shows that the canonical generator of inertia is σ of ξ is the specialization of ∞ and σ^2 otherwise. Now let ξ be a singularity of \bar{X} , and denote the irreducible components intersecting in ξ by \bar{X}_1 and \bar{X}_2 . Then the canonical generators g_i of the restrictions $\bar{\phi}|_{\bar{X}_i}$ at ξ satisfy

$$g_1 = g_2^{-1}.$$

(This last condition says that $\bar{\phi}$ is an admissible cover.)

The upshot is that the map $\bar{\phi} : \bar{Y} \rightarrow \bar{X}$ is completely determined and easily described by the stably marked curve (\bar{X}, \bar{D}) .

The following lemma lists the 5 possibilities for \bar{X} . Note that we need to distinguish between ∞ and the other 4 branch points. The proof is elementary, and therefore omitted.

Lemma 4.1. *With assumptions and notations as in the beginning of the section, we have the following 5 possibilities for \bar{X} .*

- (a) *The curve \bar{X} is irreducible.*
- (b) *The curve \bar{X} consists of two irreducible components \bar{X}_1 and \bar{X}_2 . Three of the branch points of ϕ including ∞ specialize to \bar{X}_1 , the other two to \bar{X}_2 .*
- (c) *The curve \bar{X} consists of three irreducible components \bar{X}_1 , \bar{X}_2 , and \bar{X}_3 , where \bar{X}_1 and \bar{X}_3 intersect \bar{X}_2 . The branch point ∞ specializes to \bar{X}_2 , two other branch points specialize to \bar{X}_1 , and two to \bar{X}_3 .*
- (d) *The curve \bar{X} consists of two irreducible components \bar{X}_1 and \bar{X}_2 . Three of the branch points of ϕ different from ∞ specialize to \bar{X}_1 , the other two to \bar{X}_2 .*
- (e) *The curve \bar{X} consists of three irreducible components \bar{X}_1 , \bar{X}_2 , and \bar{X}_3 , where \bar{X}_1 and \bar{X}_3 intersect \bar{X}_2 . Two branch points including ∞ specialize to \bar{X}_1 , two other branch points specialize to \bar{X}_3 , and the last one to \bar{X}_2 .*

The following result immediately follows from the possibilities for \bar{X} , together with the fact that $\bar{\phi}$ is an admissible cover.

Theorem 4.2. *Let K be as in §2.1, with $p \neq 3$. Let Y be a Picard curve over K , L/K a finite Galois extension over which Y has semistable reduction. Let \mathcal{Y} denote the stable model of Y_L over \mathcal{O}_L and $\bar{Y} := \mathcal{Y} \otimes k$ the special fiber. Then \bar{Y} is as in one of the following five cases.*

- (a) *The curve \bar{Y} is smooth.*

- (b) The curve \bar{Y} consists of two irreducible components, of genus 2 and 1, which intersect in a unique singular point.
- (c) The curve \bar{Y} has three irreducible components W_1, W_2, W_3 which are each smooth of genus 1. There are two singular points where W_1 (resp. W_3) intersects W_2 .
- (d) There are two irreducible components W_1, W_2 of genus 0 and 1, respectively, and three singular points where W_1 and W_2 intersect.
- (e) There are three irreducible components W_1, W_2, W_3 , of genus 0, 0 and 1, respectively, and 4 singular points. The components W_1, W_2 meet in three of these singular points, while W_2 and W_3 meet in the fourth.

4.2 The conductor exponent in the tame case

In the tame case, there are no useful lower bounds for the conductor exponent. In particular, Y may have good reduction in which case we have $f_{\mathfrak{p}} = 0$. Also, unlike for $p = 3$, nothing is gained by assuming that the ground field K is totally unramified. Still, some useful restrictions on $f_{\mathfrak{p}}$ can be proved (see Theorem 4.4 below).

We start by recalling a well known criterion for good reduction, see e.g. [8], § 7.

Let

$$Y : y^3 = f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

be a Picard curve over K . Replacing (x, y) by $(a_4^{-1}x, a_4^{-1}y)$ and multiplying both sides of the defining equation by a_4^3 , we may assume that $a_4 = 1$. Let $\Delta(f) \in K^\times$ denote the discriminant of f . (Since we assume that f is separable, we have $\Delta(f) \neq 0$.) After replacing (x, y) by $(u^{-3}x, u^{-4}y)$ and multiplying by u^{12} on both sides, for a suitable $u \in K^\times$, we may further assume that all coefficients $a_i \in \mathcal{O}_K$ are integral. In particular, it follows that $\Delta(f) \in \mathcal{O}_K$. Since

$$\Delta(u^{12}f(u^{-3}x)) = u^{36}\Delta(f),$$

by the right choice of u , we may assume that

$$0 \leq \text{ord}_{\mathfrak{p}}(\Delta(f)) < 36. \quad (29)$$

Lemma 4.3. *Assume that the Picard curve Y is given by a minimal equation over \mathcal{O}_K , as above. Then Y has good reduction if and only if $\Delta(f) \in \mathcal{O}_K^\times$.*

Proof. See [8], Lemma 7.13. □

Note that the forwards direction of Lemma 4.3 also follows from Theorem 4.2. Here is what we can say in general about the conductor exponent.

Theorem 4.4. *Let K be as before, with $p \neq 3$, and Y a Picard curve over K . Let $f_{\mathfrak{p}}$ denote the conductor exponent for Y , relative to the prime ideal \mathfrak{p} of \mathcal{O}_K . Then the following holds.*

- (a) If $f_{\mathfrak{p}} = 0$ then the stable reduction of Y is as in Case (a), (b), or (c) of Theorem 4.2. Furthermore, the splitting field L_0/K of f is unramified at \mathfrak{p} .

- (b) If $p = 2$ then $f_p \neq 1$.
(c) If $p \geq 5$ then $f_p \in \{0, 2, 4, 6\}$.

Proof. We start by proving Statement (a). Note that $f_p = 0$ if and only if $\delta = 0$ and $\dim H_{\text{et}}^1(\bar{Y}^0, \mathbb{Q}_\ell) = 6$. The second condition, together with the discussion after Proposition 2.3, implies that $\gamma(\bar{Y}^0)$. Statement (a) now follows immediately from Theorem 4.4.

Claim: The integer ε , defined in Proposition 2.3, is even. The discussion following Proposition 2.3 implies that f_p is odd if and only if $\dim H^1(\Delta_{\bar{Y}^0})$ is odd. The case distinction in Theorem 4.2 implies that $\dim H^1(\Delta_{\bar{Y}^0})$ is at most 2. Therefore to prove the claim, it suffices to show that $\gamma(\bar{Y}^0) = \dim H^1(\Delta_{\bar{Y}^0}) \neq 1$. We prove this in the case that \bar{Y} is as in (d) of Theorem 4.2. The argument in the case that \bar{Y} is as in (e) is very similar. In the other cases there is nothing to prove.

Assume that \bar{Y} is as in (d) of Theorem 4.2. Then \bar{X} is as (d) of Lemma 4.1 and $\bar{\phi}$ maps W_i to \bar{X}_i . Since ∞ is K -rational, the monodromy group Γ fixes it. It follows that Γ acts on the component \bar{X}_2 to which ∞ specializes. (This is similar to the argument in the proof of [2], Lemma 5.4.) Since there is exactly one other branch point specializing to \bar{X}_2 , this point is fixed by Γ , as well. Similarly, Γ fixes the unique singularity. Since Γ fixes at least 3 points on the genus-0 curve \bar{X}_2 , it acts trivially on \bar{X}_2 . Equation (2) implies that the action of Γ on \bar{Y} descends to \bar{X} . It follows that Γ acts on W_2 via a subgroup of G . We conclude that Γ either fixes the three singularities of \bar{Y} or cyclically permutes them. It follows that $\gamma(\bar{Y}^0)$ is 2 or 0. This proves the claim.

Assume that $p = 2$. Using Equation (6) one shows that if $\delta \neq 0$ then $\delta \geq 2$. Therefore Statement (b) follows from the claim.

For Statement (c) recall that L/K is at most tamely ramified for $p \geq 5$. It follows that $\delta = 0$, and hence that $f_p = \varepsilon$ is bounded by $2g(Y) = 6$. Statement (c) now follows from the claim. \square

Remark 4.5. (a) The condition $f_p = 0$ in Theorem 4.4.(a) is equivalent to the condition that the Jacobian variety of Y has good reduction over K . This is the case if and only if Y has stable reduction already over K , and the graph of components $\Delta_{\bar{Y}}$ is a tree. This observation is similar to the statement of Lemma 2.4.

(b) For $p = 2$ the conductor exponent f_2 may be odd. An example can be found in Example 5.5.

(c) The bound on f_p for $p = 5, 7$ in Theorem 4.4.(c) is slightly sharper than the bound for f_p for general abelian varieties of dimension 3 from [3], Thm. 6.2. The reason is that Brumer and Kramer obtain an upper bound for δ . For Picard curves and $p = 5, 7$ we have $\delta = 0$, whereas this is not necessarily the case for general curves of genus 3.

For $p = 2$ the result of [3] yields the upper bound $f_p \leq 28$. Distinguishing the possibilities for the stable reduction and combining our arguments with those of [3] it might be possible to improve the bound in this case.

Example 4.6. Consider the Picard curve

$$Y : y^3 = f(x) = x^4 + 14x^2 + 72x - 41$$

over $K := \mathbb{Q}_5^{\text{nr}}$. We claim that Y has semistable reduction over K , and that the reduction type is as in Case (b) of Theorem 4.2. Therefore, $f_5 = 0$.

We will argue in a similar way as in § 3.3, see in particular Example 3.9, see also [2], § 6 and § 7. The first observation is that

$$f = x^4 + 14x^2 + 72x - 41 \equiv (x+3)^2(x^2 + 4x + 1) \pmod{5}. \quad (30)$$

By Hensel's Lemma, f has two distinct roots $\alpha_1, \alpha_2 \in \mathcal{O}_K$ with $\alpha_i^2 + 4\alpha_i + 1 \equiv 0 \pmod{5}$. The other two roots of f are congruent to $-3 \pmod{5}$. Substituting $x = -58 + 5^3x_1$ into f , we see that

$$f \equiv 5^6(3x_1^2 + 4x_1 + 2) \pmod{5^7}. \quad (31)$$

It follows that f has two more roots $\alpha_3, \alpha_4 \in K$ of the form $\alpha_i = -58 + 5^3\beta_i$, with $\beta_i \in \mathcal{O}_K$ and $3\beta_i^2 + 4\beta_i + 2 \equiv 0 \pmod{5}$. So f splits over K .

Let $(\mathcal{X}, \mathcal{D})$ be the stably marked model of (X, D) , where $X = \mathbb{P}_K^1$ and $D = \{\infty, \alpha_1, \dots, \alpha_4\}$. The calculation of the α_i above show that \mathcal{X} is the \mathcal{O}_K -model of X corresponding to the set of valuations $\{v_0, v_1\}$, where v_0 (resp. v_1) is the Gauss valuation on $K(x)$ with respect to the parameter x (resp. to x_1). Let \mathcal{Y} be the normalization of \mathcal{X} in the function field of Y . We claim that the special fiber \bar{Y} of \mathcal{Y} consists of two irreducible components W_0, W_1 of geometric genus 2 and 1, respectively. By the same argument as in Example 3.9, this already implies that \mathcal{Y} is semistable and that the special fiber is as in Case (b) of Theorem 4.2.

To prove the claim it suffices to find generic equations for W_0 and W_1 . For W_0 we just have to reduce the original equation for Y modulo 5. By (30) we obtain

$$W_0 : \bar{y}^3 = (\bar{x} + 3)^2(\bar{x}^2 + 4\bar{x} + 1),$$

which shows that $g(W_0) = 2$. For W_1 we write f as a polynomial in x_1 , substitute $y = 5^2w$, divide by 5^6 and reduce modulo 5. By (31) we obtain

$$W_1 : \bar{w}^3 = 3\bar{x}_1^2 + 4\bar{x}_1 + 2,$$

which shows that $g(W_1) = 1$. Now everything is proved. \square

Remark 4.7. The example above is again rather special, since $f_5 = 0$ even though Y has bad reduction at $p = 5$. (See also Definition 5.4).

5 Searching for Picard curves over \mathbb{Q} with small conductor

In this last section we briefly address the problem of constructing Picard curves with small conductor. We think this is an interesting problem which deserves further investigation. The main background result here is the *Shafarevic conjecture* (which is a theorem due to Faltings). We use this theorem via the following corollary.

Theorem 5.1 (Faltings). *Fix a number field K and an integer $g \geq 2$.*

- (a) For any finite set S of finite places of K there exist at most a finite number of isomorphism classes of smooth projective curves of genus g over K with good reduction outside S .
- (b) For any constant $N > 0$ there exists at most a finite number of isomorphism classes of curves of genus g over K with conductor $\leq N$.

Proof. Satz 6 in [6] states that there are at most a finite number of d -polarized abelian varieties of dimension g over K with good reduction outside S , for fixed K , g , d and S . Statements (a) and (b) follow from this. For (a), one simply uses Torelli's theorem (see [6], p. 365, Korollar 1). To deduce (b) we use that the conductor of a curve Y is the same as the conductor of its Jacobian, and that an abelian variety over K has bad reduction at a finite place \mathfrak{p} of K if and only if $f_{\mathfrak{p}} = 0$ (see e.g. [23], Theorem 1). \square

Unfortunately, no effective proof of Theorem 5.1 is known in general.² However, for some special classes of curves effective proofs are known, see e.g. [10].

The problem we wish to discuss here is whether the statement of Theorem 5.1 can be made computable in the case of Picard curves. More precisely: given a finite set S of rational primes (or a bound $N > 0$), can we compute the finite set of curves with good reduction outside S (resp. with conductor $\leq N$)? Note that this is not equivalent to (and may be much easier than) having an effective proof of Theorem 5.1 for Picard curves. For the first problem, the answer is known to be affirmative:

Proposition 5.2. *There exists an algorithm which, given as input a number field K and finite set S of finite places of K , computes the set of isomorphism classes of all Picard curves Y/K with good reduction outside S .*

Proof. This is an adaption to Picard curves of the algorithm given by Smart for hyperelliptic curves, see [25] and [13]. The idea is that it suffices to determine the finite set of equivalence classes of binary forms of degree 4 over K whose discriminant is an S -unit (corresponding to the polynomial $f(x)$). The latter problem can be reduced to solving an S -unit equation, for which effective algorithms are known. \square

Example 5.3. Let $K = \mathbb{Q}$ and $S = \{3\}$. Then there are precisely 63 isomorphism classes of Picard curves over \mathbb{Q} with good reduction outside S . See [13].

For example, the curve

$$Y : y^3 = f(x) = x^4 - 3x^3 - 24x^2 - x$$

has good reduction outside $S = \{3\}$ (the discriminant of f is $\Delta(f) = 3^{10}$). The stable reduction \bar{Y} of Y at $p = 3$ is as in Case (c) of Theorem 3.2, the exponent conductor is $f_3 = 10$ (see [4], Appendix A1.1). This is the lowest value for the conductor which occurs for the curves in the list of [13]. The conductor exponents of all 63 Picard curves from [13] have been computed in [4], Appendix A1.2. From

² The precise meaning of an *effective proof* is that it provides an explicitly computable bound on the height of the curve or abelian variety in question.

this calculation it follows that the conductor exponent f_3 only takes the values $f_3 = 10, 11, 12, 13, 15, 17, 19, 21$.

The upper bound on the conductor exponent from abelian varieties of genus 3 from [3], Theorem 6.2 yields $f_3 \leq 21$. The result stated above therefore implies that this bound is also obtained for Picard curves.

Unfortunately we do not know any algorithm for solving (b), i.e. for finding all Picard curves with bounded conductor. The reason that the method for (a) does not solve (b) is the existence of *exceptional primes*.

Definition 5.4. Let Y be a Picard curve over \mathbb{Q} and p a prime number. Then p is called *exceptional* with respect to Y if Y has bad reduction at p and $f_p = 0$.

Exceptional primes are rather rare. It can easily be shown, using the arguments from this paper, that if p is a exceptional prime for Y then the splitting field of the polynomial f is unramified at p , and

$$\text{ord}_p(\Delta(f)) \in \{6, 12\}.$$

Example 5.5. We consider the Picard curve over \mathbb{Q}

$$Y : y^3 = f(x) = x^4 + 14x^2 + 72x - 41.$$

The discriminant of f is $\Delta(f) = -2^{10}3^45^6$. So Y has good reduction outside $S = \{2, 3, 5\}$. We have shown in Example 4.6 that $f_5 = 0$, i.e. that 5 is an exceptional prime. Using the methods of [2] and [18] one can prove that $f_2 = 19$ and $f_3 = 13$ (see e.g. this SageMathCloud worksheet: <http://tinyurl.com/hp3qzmo>, [20]). All in all, the conductor of Y is

$$N_Y = 2^{19}3^{13} = 835884417024.$$

Although S is small and $p = 5$ is an exceptional prime, N_Y is relatively large. We have tried but were not able to find a similar example with exceptional primes and a significantly smaller conductor. Nevertheless, the fact that exceptional primes exist means that we cannot easily bound the size of the set S while searching for Picard curves with bounded conductor.

Here is an example of a Picard curve with a relatively small conductor.

Example 5.6. Consider the Picard curve

$$Y/\mathbb{Q} : y^3 = f(x) = x^4 - 1.$$

The discriminant of f is $\Delta(f) = -256 = -2^8$. It follows that Y has good reduction outside $S = \{2, 3\}$. By [4], § 5.1.3, we have $f_2 = 6$ and $f_3 = 6$. Therefore,

$$N_Y = 2^63^6 = 46656.$$

The first named author has made an extensive search for Picard curves over \mathbb{Q} with small conductor ([4], § 5.3). Among all computed examples, the curve Y was the one with the smallest conductor.

A remarkable property of the curve Y is that for every (rational) prime p it admits a map to \mathbb{P}^1 of order prime to p , which becomes Galois over an extension: besides the degree-3 map ϕ given by $(x, y) \mapsto x$, we have the map $(x, y) \mapsto y$, which has degree 4. In fact, the full automorphism group of Y has order 48, and is maximal in the sense that $Y/\text{Aut}_{\mathbb{C}}(Y)$ is a projective line, and the natural cover is branched at three points.

It is instructive to compare the above example with the curve

$$Y' : y^3 = x^4 + 1.$$

This is a twist of Y . The curve Y and Y' become isomorphic over $\mathbb{Q}[i]$, yet have different conductors. In fact,

$$N_{Y'} = 2^{16}3^6,$$

see [4], § 5.1.2.

We propose to study the following problem.

Problem 5.7. Prove that the curve from Example 5.6 is the only Picard curve (up to isomorphism) with conductor $N_Y \leq 46656$, or find explicit counterexamples.

Proposition 5.2 and our main results (Theorem 3.6 and Theorem 4.4) suggest the following strategy for construction Picard curves with small conductor and thereby finding counterexamples. If we ignore the possibility of exceptional primes, a Picard curve with conductor $\leq 2^{63}3^6$ must have good reduction outside S , where S is one of the following sets:

- $\{2, 3, p\}$, $p \leq 13$,
- $\{3, p\}$, $p \leq 23$.

To find all such curves looks challenging but within reach. It should also be very useful to take into account the local restrictions on the polynomial f imposed by our results on curves with a specific value for f_p . On the other hand, without an effective proof of Theorem 5.1 (b) for Picard curves, it is not clear at the moment how one could actually prove that the curve from Example 5.6 (or any other curve we may find) has minimal conductor.

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